Available at WWW.MATHEMATICSWEB.ORG POWERED BY SCIENCE DIRECT®

ELSEVIER

Stochastic Processes and their Applications 107 (2003) 213–231

stochastic processes and their applications

www.elsevier.com/locate/spa

A multidimensional bipolar theorem in $L^0(\mathbb{R}^d;\Omega,\mathscr{F},P)$

B. Bouchard*, L. Mazliak

Laboratoire de Probabilités et Modèles aléatoires, UMR CNRS 7599 et Université Paris VI, boîte courrier 188, 4 place Jussieu, 75252 Paris Cedex 05, France

Received 7 February 2002; received in revised form 30 April 2003; accepted 12 May 2003

Abstract

In this paper, we prove a multidimensional extension of the so-called Bipolar Theorem proved in Brannath and Schachermayer (Séminaire de Probabilités, vol. XXX, 1999, p. 349), which says that the bipolar of a convex set of positive random variables is equal to its closed, solid convex hull. This result may be seen as an extension of the classical statement that the bipolar of a subset in a locally convex vector space equals its convex hull. The proof in Brannath and Schachermayer (ibidem) is strongly dependent on the order properties of $\mathbb{R}$. Here, we define a (partial) order structure with respect to a d-dimensional convex cone K of the positive orthant $[0,\infty)^d$. We may then use compactness properties to work with the first component and obtain the result for convex subsets of K-valued random variables from the theorem of Brannath and Schachermayer (ibidem). As a byproduct, we obtain an equivalence property for a class of minimization problems in the spirit of Kramkov and Schachermayer (Ann. Appl. Probab 9(3) (1999) 904, Proposition 3.2). Finally, we discuss some applications in the context of duality theory of the utility maximization problem in financial markets with proportional transaction costs.

MSC: 46N30; 46A20; 46A16; 06A06

Keywords: Polarity; Convex analysis; Partial order; Convergence in probability; Dual formulation in mathematical finance

1. Introduction

Let $(\Omega,\mathscr{F},P)$ be a probability space and denote by $L^0(K;\Omega,\mathscr{F},P)$ the linear space of equivalent classes of K-valued random variables (with $K\subset\mathbb{R}^d$) on $(\Omega,\mathscr{F},P)$,

* Corresponding author. Tel.: +33-1-44-27-39-73; fax: 33-1-44-27-72-23.
E-mail address: bouchard@ccr.jussieu.fr (B. Bouchard).

0304-4149/03/$ - see front matter © 2003 Elsevier B.V. All rights reserved.
doi:10.1016/S0304-4149(03)00073-5

equipped with the topology of convergence in measure. Although this space fails to be locally convex, it was shown in Brannath and Schachermayer (1999), that an analogue to the classical bipolar theorem can be obtained for subsets of $L^0(\mathbb{R}_+;\Omega,\mathscr{F},P)$, if we put $L^0(\mathbb{R}_+;\Omega,\mathscr{F},P)$ in duality with itself, the 'scalar product' $\langle Y,Z\rangle := E[YZ]$ taking possibly infinite values. The polar of a subset C of $L^0(\mathbb{R}_+;\Omega,\mathscr{F},P)$ is then defined as $C^0 := \{X\in L^0(\mathbb{R}_+;\Omega,\mathscr{F},P): E[XY]\leqslant 1 \text{ for all } Y\in C\}$ and the bipolar $C^{00} := \{Y\in L^0(\mathbb{R}_+;\Omega,\mathscr{F},P): E[XY]\leqslant 1 \text{ for all } X\in C^0\}$ is characterized as being the smallest closed (in measure), convex and solid subset of $L^0(\mathbb{R}_+;\Omega,\mathscr{F},P)$ containing C. Here, the order structure of $\mathbb{R}$ and the notion of solidity plays a crucial role. It means that for $Y\in C^{00}$ and $Z\in L^0(\mathbb{R}_+;\Omega,\mathscr{F},P)$, if $Y\geqslant Z$, then $Z\in C^{00}$. In particular, if C is convex, the above characterization implies that any element $Z\in C^{00}$ can be approximated, from above, by a sequence in C, in the sense that for any $Z\in C^{00}$ there exists a sequence $(Y_n)_n$ in C such that $Y_n\to_P Y\geqslant Z$. It follows that, roughly speaking, minimizing a non-increasing convex functional over C or C^{00} are equivalent, whenever C is convex. As C^{00} is closed in probability, it will be generally easier to obtain the existence of an extremum in C^{00} but this will not change the associated value function.

The formulation of this bipolar result was originally motivated by an application in Mathematical Finance. In the theory of financial markets, it is usual to rely on a duality relation between some random variables, interpreted as contingent claims, and on Radon–Nicodym derivatives of some absolutely continuous martingale measures. As $L^0(\mathbb{R}_+;\Omega,\mathscr{F},P)$ remains unchanged if P is replaced by an equivalent measure, it turns out to be a natural space to work on (see e.g. Delbaen and Schachermayer (1999), for a description of these duality relations).

However, the above mentioned theory does not allow to take into account the costs arising from transactions when dealing with a financial market. In models with transaction costs, each component of a portfolio process needs to be isolated as it corresponds to an investment in a particular asset. This means that a portfolio must be modeled by a vector valued process. This problem does not appear in frictionless markets where exchanges can be instantaneously performed without any impact on the global wealth value. Similarly, contingent claims are modeled by $\mathbb{R}^d$-valued random variables, each component corresponding to a liability in a given financial asset. Finally, it turns out that Radon–Nikodym derivatives need to be replaced by a family of processes with values in a convex cone K of $[0,\infty)^d$. It follows that the natural space to work on is $L^0(K;\Omega,\mathscr{F},P)$ instead of $L^0(\mathbb{R}_+;\Omega,\mathscr{F},P)$ (see Section 3.2 and the references therein).

Motivated by the above discussion, we extend the bipolar theorem of Brannath and Schachermayer (1999), to convex subsets of $L^0(K;\Omega,\mathscr{F},P)$, where K is a convex cone of $[0,\infty)^d$. A natural way to do this consists in placing $L^0(K;\Omega,\mathscr{F},P)$ in polarity with $L^0(K^0;\Omega,\mathscr{F},P)$, where K^0 is the positive polar of K in the sense of convex analysis. Considering the partial ordering induced by K, we then introduce a notion of K-solidity which extend the notion of solidity defined in Brannath and Schachermayer (1999), for subsets of $L^0(\mathbb{R}_+;\Omega,\mathscr{F},P)$. This result is then applied to the duality theory of utility maximization in financial markets with proportional transaction costs.

2. The bipolar theorem

To a closed convex cone K of $[0,\infty)^d$, i.e. that for $x\in K$ and $\lambda\geqslant 0$, $\lambda x\in K$, one may associate its *positive polar*

$$K^0 := \left\{ x\in\mathbb{R}^d : x\cdot y := \sum_{i=1}^d x^i y^i \geqslant 0 \ \forall y\in K \right\}.$$

One may easily prove that $K^{00}=K$ as a consequence of the Hahn–Banach theorem.

Given the closed convex cone K in $[0,\infty)^d$, one may define a partial order $\succcurlyeq_K$ on $[0,\infty)^d$ by the following relation

$$y \succcurlyeq_K y' \quad\Leftrightarrow\quad y-y'\in K,$$

which, by means of the Hahn–Banach theorem, can be equivalently defined by

$$y \succcurlyeq_K y' \quad\Leftrightarrow\quad x\cdot(y-y')\geqslant 0 \text{ for all } x\in K^0.$$

We now consider a corresponding notion for a convex subset C of $L^0(K;\Omega,\mathscr{F},P)$, the set of K-valued random variables. For such a set, we define its "positive" polar C^0 as

$$C^0 := \{X\in L^0(K^0;\Omega,\mathscr{F},P) : E[X\cdot Y]\leqslant 1 \text{ for all } Y\in C\}.$$

The bipolar C^{00} of C is then defined as

$$C^{00} := \{Y\in L^0(K;\Omega,\mathscr{F},P) : E[X\cdot Y]\leqslant 1 \text{ for all } X\in C^0\}.$$

On $L^0(K;\Omega,\mathscr{F},P)$, we shall work with the topology of convergence in probability. In particular, we shall say that a subset of $L^0(\mathbb{R}^d;\Omega,\mathscr{F},P)$ is *closed* if it is closed in probability.

Definition 2.1. Let K be a convex cone of $[0,\infty)^d$ and C a subset of $L^0(K;\Omega,\mathscr{F},P)$ We say that

(i) C is K-solid if

$$f\in C,\ g\in L^0(K;\Omega,\mathscr{F},P) \text{ and } f\succcurlyeq_K g\, P\text{-a.s.} \Rightarrow g\in C.$$

(ii) C is bounded in probability, if for $\varepsilon>0$, we can find some $M>0$ such that:

$$P[\|f\|>M]<\varepsilon \quad \text{for all } f\in C$$

where $\|.\|$ denotes the euclidian norm on $\mathbb{R}^d$.

(iii) C is hereditarily unbounded on a set $A\in\mathscr{F}$ if, for every $B\subset A$ with $P(B)>0$, its restriction to B, $\{f\mathbb{1}_B,\ f\in C\}$, fails to be bounded in probability.

From now on, we consider K a closed convex cone of $[0,\infty)^d$ and C a convex subset of $L^0(K;\Omega,\mathscr{F},P)$. In the following, we denote by $\mathscr{K}$ the linear subspace generated by K.

Throughout, we shall use the following hypotheses.

H1: $K \cap \partial[0,\infty)^d = \{0\}$.
H2: $C \cap L^0(\mathrm{ri}(K);\Omega,\mathscr{F},P) \neq \emptyset$, where $\mathrm{ri}(K)$ stands for the relative interior of K as a subset of $\mathscr{K}$.

We are now in position to formulate an extension of the unidimensional Bipolar Theorem set in Brannath and Schachermayer (1999).

Theorem 2.2. *Let K be a closed convex cone in $[0,\infty)^d$ and C be a convex subset of $L^0(K;\Omega,\mathscr{F},P)$. Then*

(i) *C^0 is closed, convex and K^0-solid.*
(ii) *Assume further that* H1–H2 *hold. Let D be the smallest closed, convex, K-solid set in $L^0(K;\Omega,\mathscr{F},P)$ containing C. Then, $C^{00} = D$.*
(iii) *Let $\bar{C}$ denote the closed hull of C for the convergence in probability. Then,*

$$D = \{f \in L^0(K;\Omega,\mathscr{F},P),\ \exists Y \in \bar{C},\ Y \succcurlyeq_K f\}.$$

Remark 2.3. It follows from (iii) that for all $f \in C^{00}$, there exists some sequence $(Y_n) \in C$ such that $Y_n \to Y \in C^{00}$ in probability with $Y \succcurlyeq_K f$.

The proof is based on an extension of Lemma 2.3 in Brannath and Schachermayer (1999) which considers the case $d = 1$ and $K = \mathbb{R}_+$. Before setting this result and its proof, let us make some technical remarks about the topology of the convex set K and it positive polar K^0. We set $\tilde{K}^0 := K^0 \cap \mathscr{K}$.

Property 2.4. *Let K be a closed convex cone in $[0,\infty)^d$ satisfying* H1. *Fix some vector $e \in \mathrm{ri}(K)$ with $\|e\| = 1$.*

(i) *$K \subset [0,\infty)^d \subset K^0$ so that $K \subset \tilde{K}^0$.*
(ii) *Let $\tilde{K}^0_1 := \tilde{K}^0 \cap \{x \in \mathbb{R}^d : \|x\|_e := \sum_{i=1}^d |x^i|e^i = 1\}$. Then, $\ell_K(y) := \min_{x \in \tilde{K}^0_1} x \cdot y \leqslant \|y\|_e$ for all $y \in \mathbb{R}^d$.*
(iii) *For any $y \in K$ and any $\lambda > 0$, $\ell_K(\lambda y) = \lambda \ell_K(y)$. Moreover, $\ell_K(.)$ is continuous on K.*
(iv) *For $y \in \mathscr{K}$, $y \in K \ \Leftrightarrow\ \ell_K(y) \geqslant 0$.*
(v) *For $y \in \mathscr{K}$, $y \in \mathrm{ri}(K) \ \Leftrightarrow\ \ell_K(y) > 0$.*
(vi) *For $y \in K$, one has $y \succcurlyeq_K \ell_K(y)e$ and $-\ell_K(-y)e \succcurlyeq_K y\ell_K(e)$.*
(vii) *There is a constant $c_e > 0$ such that $\ell_K(y) \geqslant c_e f$, for all $y \in K$ and $f \in \mathbb{R}_+$ such that $y \succcurlyeq_K fe$.*

The proof of the above list is given in Appendix A.

We now set the required extension of Lemma 2.3 in Brannath and Schachermayer (1999).

Lemma 2.5. *Let $K \subset [0,\infty)^d$ be a convex cone satisfying* H1, $e \in \mathrm{ri}(K)$ *such that* $\|e\| = 1$, *and C be a convex subset of $L^0(K;\Omega,\mathscr{F},P)$ satisfying* H2. *Then, there exists a partition of Ω into disjoint sets Ω_u, $\Omega_b \in \mathscr{F}$ such that*

(i) *The restriction C_b of C to Ω_b is bounded in probability.*
(ii) *C is hereditarily unbounded in probability on Ω_u.*
(iii) *If $P[\Omega_b] > 0$, there exists a probability measure $Q_b \sim P|_{\Omega_b}$ such that C is bounded in $L^1(\mathbb{R}^d;\Omega,\mathscr{F},Q_b)$. Moreover, we can choose Q_b such that dQ_b/dP is uniformly bounded.*
(iv) *For all $\varepsilon > 0$, there exists some f in*

$$\tilde{C} := \{f \in L^0(\mathbb{R}_+;\Omega,\mathscr{F},P) : \exists Y \in C,\ Y \succcurlyeq_K fe\}.$$

such that

$$P[\Omega_u \cap \{f < \varepsilon^{-1}\}] < \varepsilon.$$

(v) *Letting D be the smallest closed, convex, K-solid set in $L^0(K;\Omega,\mathscr{F},P)$ containing C, then*

$$D = D_b \oplus L^0(K;\Omega,\mathscr{F},P)|_{\Omega_u},$$

where D_b is the restriction of D to Ω_b.

Proof. (1) We shall use Lemma 2.3 of Brannath and Schachermayer (1999), to construct the two sets Ω_b and Ω_u. Define

$$\tilde{C} := \{f \in L^0(\mathbb{R}_+;\Omega,\mathscr{F},P) : \exists Y \in C,\ Y \succcurlyeq_K fe\}.$$

It is obviously a convex subset of $L^0(\mathbb{R}_+;\Omega,\mathscr{F},P)$ and therefore, by Lemma 2.3 of Brannath and Schachermayer (1999), there exist Ω_b and Ω_u such that the restriction of $\tilde{C}$ to Ω_b is bounded in probability, $\tilde{C}$ is hereditarily unbounded in probability on Ω_u, and (iv) holds.

(2) We now prove (ii). Let $\varepsilon > 0$ and B a measurable subset of Ω_u such that $P(B) > 0$. By Lemma 2.3 of Brannath and Schachermayer (1999), for each $M > 0$, there exists some $f \in \tilde{C}$ such that $P[B \cap \{f > M/c_e\}] \geqslant \varepsilon$. By definition of $\tilde{C}$, there is $Y \in C$ such that $Y \succcurlyeq_K fe$ and so, by Property 2.4 (ii) and (vii), $\|Y\|_e \geqslant \ell_K(Y) \geqslant c_e f$. Therefore, $P(B \cap \{\|Y\|_e > M\}) \geqslant P[B \cap \{f > M/c_e\}] \geqslant \varepsilon$.

(3) We prove (i) and (iii). Observe that $\{y \in K : y^1 = 1\}$ is compact since $K \cap \partial[0,\infty)^d = \{0\}$. Therefore, using Property 2.4 (iii) and (v), there exists some $c_K > 0$ such that

$$\ell_K(y) \geqslant c_K y^1 \quad \text{for all } y \in \mathrm{ri}(K). \tag{2.1}$$

Fix $\bar{Y}$ in $L^0(\mathrm{ri}(K);\Omega,\mathscr{F},P) \cap C$. For all $Y \in C$, $\frac{1}{2}(Y + \bar{Y}) \in L^0(\mathrm{ri}(K);\Omega,\mathscr{F},P) \cap C$, so that $\ell_K(\frac{1}{2}(Y + \bar{Y})) \geqslant c_K/2(Y^1 + \bar{Y}^1)$. As $\{y \in K : y^1 = 1\}$ is compact, there is an $M > 0$ such that for all $y \in K$, $\|y\| \leqslant Mc_K/2y^1$ and so

$$\|Y\| \leqslant M\frac{c_K}{2}Y^1 \leqslant M\frac{c_K}{2}(Y^1 + \bar{Y}^1) \leqslant M\ell_K\left(\frac{1}{2}(Y + \bar{Y})\right). \tag{2.2}$$

Now, from Property 2.4 (vi), $\{\ell_K(Y),\, Y\in C\}\subset\tilde C$ so that $\ell_K(\frac{1}{2}(Y+\bar Y))\in\tilde C$. As $\tilde C$ is bounded in probability on Ω_b, one obtains (i) by (2.2). (iii) is obtained in the same way, by application of Lemma 2.3 (3) of Brannath and Schachermayer (1999), to $\tilde C$.

(4) We finally prove (v). Observing that the set D will not change if we replace C by its K-solid hull, we shall assume from now on that C is K-solid.

Clearly, $D\subset D_b\oplus L^0(K;\Omega,\mathscr F,P)|_{\Omega_u}$. We prove the inverse inclusion. Let $h=v+w$ for $v\in D_b$ and $w\in L^0(K;\Omega,\mathscr F,P)|_{\Omega_u}$.

For $f\in\tilde C$, there exists $Y\in C$ such that $Y\succcurlyeq_K fe$. Therefore, using Property 2.4 (vii), $\ell_K(Y)\geqslant fc_e$ and so $Y\succcurlyeq_K\ell_K(Y)e\succcurlyeq_K fc_ee$, the first relation coming from Property 2.4 (vi). As C is K-solid, $fc_ee\in C$. We have proved that

$$\{c_efe: f\in\tilde C\}\subset C.$$

Using (iv), we therefore can choose a sequence $f_ne\in C$ such that

$$P[\Omega_u\cap\{f_n<n^2\}]<1/n^2<1/n.$$

Let $A_n:=\{f_ne\succcurlyeq_K nw\}$ and define $w_n=nw\mathbb{1}_{A_n}+f_ne\mathbb{1}_{A_n^c}$.

Set $h_n:=(1-1/n)v+(1/n)w_n$. By definition of A_n, $f_ne\succcurlyeq_K w_n$ and so, by K-solidity of C, $w_n\in C$. Therefore, $w_n\in D$. As $v\in D$, and D is convex, $h_n\in D$. We claim that

$$\frac{1}{n}\,w_n\to w \quad \text{in probability as } n\to\infty, \tag{2.3}$$

so that $h\in D$ by closedness of D.

By choice of f_n, observe that $f_n/n\to+\infty$ in probability on Ω_u. Therefore, we may replace f_n by a subsequence still denoted (f_n) for which the convergence takes place almost surely and so we may suppose that for almost every $\omega\in\Omega_u$, $f_n(\omega)/n\to+\infty$. By the homogeneity of ℓ_K (see Property 2.4 (iii)) one has

$$\ell_K(f_n(\omega)e-nw(\omega))=f_n(\omega)\ell_K\left(e-\frac{n}{f_n(\omega)}w(\omega)\right).$$

Arguing ω by ω, we see that the last quantity is nonnegative for large n, as ℓ_K is continuous, $\ell_K(e)>0$ as $e\in\mathrm{ri}(K)$, and $f_n(\omega)/n\to+\infty$. Therefore, for large n, $f_n(\omega)e-nw(\omega)\in K$ (by Property 2.4 (iv)) and so $f_n(\omega)e\succcurlyeq_K nw(\omega)$ which is to say $w_n(\omega)=nw(\omega)$. This shows (2.3). □

We now go on with the proof of our main result.

Proof of Theorem 2.2. (1) The fact that C^0 is closed is easily obtained using Fatou's Lemma and the convexity is obvious. Moreover, if $X\in C^0$ and $X\succcurlyeq_K X'$, one has that for any $Y\in K^{00}=K$, $(X-X')\cdot Y\geqslant 0$. Therefore, $X\cdot Y\geqslant X'\cdot Y$ and so $E(X'\cdot Y)\leqslant E(X\cdot Y)\leqslant 1$ hence the solidity.

(2) Let us now prove (ii). By Lemma 2.5 there exists a partition $\{\Omega_u,\Omega_b\}$ of Ω such that the restriction $C_b:=\{f\mathbb{1}_{\Omega_b}: f\in C\}$ of C to Ω_b is bounded in probability.

Let D be the smallest closed, convex, K-solid set in $L^0(K;\Omega,\mathscr F,P)$ containing C. By Lemma 2.5

$$D=D_b\oplus L^0(K;\Omega,\mathscr F,P)|_{\Omega_u}. \tag{2.4}$$

We want to prove $C^{00}=D$. As C^{00} is closed, convex and K-solid, and obviously contains C, one has $C^{00}\supset D$.

Suppose first that $P(\Omega_b)=0$. Then, $D=L^0(K;\Omega,\mathscr{F},P)\supset C^{00}$. Since $D\subset C^{00}$, one has the desired result.

We now consider the case $P[\Omega_b]>0$, so that we can find a probability measure $Q_b\sim P|_{\Omega_b}$ such that C is bounded in $L^1(\mathbb{R}^d;\Omega,\mathscr{F},Q_b)$.

Let us prove that $C^{00}\subset D$.

Suppose that we can find some $f_0\in C^{00}\setminus D$ and denote $f_b=f_0\mathbb{1}_{\Omega_b}$. Define

$$\tilde{D}_b:=\{h\in L^1(\mathbb{R}^d;\Omega,\mathscr{F},Q_b):\exists f\in D_b,\ f\succcurlyeq_K h,\ Q_b\text{-a.s.}\}.$$

Observe that this set is not empty since $C\subset D\cap L^1(\mathbb{R}^d;\Omega,\mathscr{F},Q_b)$. We now prove that $\tilde{D}_b$ is closed in probability.

Let (h_n) be a sequence in $\tilde{D}_b$ such that $h_n\to h$ in probability. By the definition of $\tilde{D}_b$, there is a sequence $(f_n)\in D_b$ such that for each $n, f_n\succcurlyeq_K h_n$. For each $i,(f_n^i)$ is a sequence of non-negative random variables so that, by Lemma 3.3 in Kramkov and Schachermayer (1999), there is a sequence $(\tilde{f}_n)$ with $\tilde{f}_n\in\operatorname{conv}(f_k,\,k\geqslant n)$ that converges P-a.s. to some $\tilde{f}$ with values in $[0,\infty]^d$. As D_b is convex and closed, we see that $\tilde{f}\in D_b$. Let $(\tilde{h}_n)$ denote the sequence of convex combinations of (h_n), constructed with the same coefficients as the $\tilde{f}_n$'s from the sequence (f_n). We have, by the convexity of K,

$$\tilde{f}_n\succcurlyeq_K\tilde{h}_n\quad\text{for each } n\geqslant 1.$$

Since, up to subsequences, $\tilde{f}_n\to\tilde{f}$ and $\tilde{h}_n\to h$ P-a.s., we get by closedness of K that $\tilde{f}\succcurlyeq_K h$ so that $h\in\tilde{D}_b$.

We can now conclude the proof of (ii). For $n\geqslant 0$, consider $f_n:=f_b\mathbb{1}_{\{\|f_b\|\leqslant n\}}$. It is an element of C^{00} by the K-solidity of C^{00}. One has $f_n\to f_b$, Q_b-a.s.

Let us prove that $f_n\in\tilde{D}_b$. Assume indeed that $f_n\in C^{00}\setminus\tilde{D}_b$. By the separation version of Hahn–Banach theorem (see e.g. Schaefer, 1996), using the fact that $\tilde{D}_b$ is closed in probability and therefore closed in L^1, there exists some $X\in L^\infty(\mathbb{R}^d;\Omega,\mathscr{F},Q_b)$, such that $E(f_n\cdot X)>1$ and $E(f\cdot X)\leqslant 1$ for all $f\in\tilde{D}_b$. As the constant function with value $0_{\mathbb{R}^d}$ is in D (by solidity), one has $L^1(-K;\Omega,\mathscr{F},Q_b)\subset\tilde{D}_b$. Define

$$Y=\arg\min_{y\in K,\,y^1=1} y\cdot X.$$

It is a bounded random variable taking values in K as by compactness and convexity the previous minimum is unique and attained continuously. Moreover, by the definition of K^0,

$$Y\cdot X\mathbb{1}_{X\notin K^0}<0. \tag{2.5}$$

For $k>0$, define $\psi_k:=-kY\mathbb{1}_{X\notin K^0}\mathbb{1}_{\Omega_b}$. This is an element of $\tilde{D}_b$. Therefore, for any $k>0$,

$$E(\psi_k\cdot X)=-kE(Y\cdot X\mathbb{1}_{X\notin K^0}\mathbb{1}_{\Omega_b})\leqslant 1.$$

As $-E(Y\cdot X\mathbb{1}_{X\notin K^0}\mathbb{1}_{\Omega_b})\geqslant 0$, this implies $-E(Y\cdot X\mathbb{1}_{X\notin K^0}\mathbb{1}_{\Omega_b})=0$ and so $-Y\cdot X\mathbb{1}_{X\notin K^0}=0$, Q_b-a.s. Using (2.5), one sees that $X\in K^0, Q_b$-a.s. For any $Y\in C$, $Y\mathbb{1}_{\Omega_b}\in\tilde{D}_b$ and therefore $1 \geqslant E(Y\mathbb{1}_{\Omega_b} \cdot X) = E(Y \cdot X\mathbb{1}_{\Omega_b})$ which means that $X\mathbb{1}_{\Omega_b} \in C^0$. As $f_n = f_n\mathbb{1}_{\Omega_b}$, one has $E(f_n \cdot X) = E(f_n \cdot X\mathbb{1}_{\Omega_b}) > 1$ and so $f_n \notin C^{00}$: a contradiction.

So $f_n \in \tilde{D}_b$. It follows that there exists some $\psi \in D_b \subset D$ such that $\psi \succcurlyeq_K f_n$, Q_b-a.s. This shows that $f_n \in D$ as $Q_b \sim P$ and D is solid.

Finally, as D is closed and $f_n \to f_b$, Q_b-a.s., $f_b \in D_b$ and by (2.4), this implies $f_0 \in D$, a contradiction. Hence the converse inclusion $C^{00} \subset D$ and the required equality.

(3) We finally prove (iii). Since D is closed and $C \subset D$, one has $\bar{C} \subset D$. As D is solid, the solid hull of $\bar{C}$,

$$\operatorname{solid}(\bar{C}) := \{f \in L^0(K; \Omega, \mathscr{F}, P),\ \exists Y \in \bar{C},\ Y \succcurlyeq_K f\}$$

is a subset of D. As $\operatorname{solid}(\bar{C})$ is moreover obviously convex, it is sufficient to prove that $\operatorname{solid}(\bar{C})$ is closed in probability to obtain the equality between the two sets. This is obtained by the same arguments as in the above proof of the closedness of $\tilde{D}_b$. □

3. Application to minimization problem

We start by defining the optimization problem in an abstract form. The interpretation in the context of financial markets with transaction costs will be discussed at the end of this section.

3.1. The general case

In all this section, we shall consider a real valued convex mapping V on $\mathbb{R}^d$ with

$$\overline{\operatorname{dom}(V)} = K, \tag{3.1}$$

where K is a closed convex cone satisfying H1 and $\operatorname{dom}(V) := \{z \in \mathbb{R}^d, |V(z)| < +\infty\}$. We further assume that V is *K-non-increasing* in the sense that

$$y \succcurlyeq y' \Rightarrow V(y') \geqslant V(y). \tag{3.2}$$

This notion of K-monotonicity is natural as we are working with the partial ordering $\succcurlyeq_K$. Finally, we suppose that there are some $b > 0$ and $\beta > 0$ such that

$$V(\lambda y) \leqslant \lambda^{-\beta} V(y), \text{ for all } \lambda \in (0,1],\ y \in \operatorname{dom}(V), \text{ with } \ell_K(y) \leqslant b. \tag{3.3}$$

Remark. The previous hypothesis is related to the following so-called *reasonable asymptotic elasticity* condition:

$$\limsup_{\ell_K(y) \to 0} \left(\sup_{q \in -\partial V(y)} q \cdot y \right) \Big/ V(y) < \infty, \tag{3.4}$$

where $\partial V(y)$ is the subgradient of V in the sense of convex analysis. The notion of reasonable asymptotic elasticity has been introduced in the context of financial markets in Kramkov and Schachermayer (1999) and appears to be necessary and sufficient for several key results in the duality theory of utility maximization (see the survey paper

Schachermayer, 2000). It has been extended to the context of non-smooth functions on $\mathbb{R}^d$ in Deelstra et al. (2002). An important property of convex functions satisfying (3.1), (3.2) and (3.4) is that there is some $b > 0$ and $\beta > 0$ such that (3.3) is satisfied (see Kramkov and Schachermayer (1999), for the one dimensional case and Lemma 4.1 in Deelstra et al. (2002), for the multivariate case).

We now consider C, a convex subset of $L^0(K;\Omega,\mathscr{F},P)$ satisfying H2. The optimization problem is defined as

$$v(C) := \inf_{Y\in C,\ E(|V(Y)|)<\infty} E[V(Y)]. \tag{3.5}$$

The usual convention $\inf\emptyset = \infty$ is used to define the above quantity.

It is well known that problem (3.5) has, in general, no optimal solution even in the case where C is bounded in $L^1(K;\Omega,\mathscr{F},P)$ (see Kramkov and Schachermayer (1999) for a counter example in a financial context), so that we need to extend our optimization problem to the closed, convex and K-solid hull of C. The following proposition shows that this can be done without changing the value function.

Proposition 3.1. *Let C be a convex subset of $L^0(K;\Omega,\mathscr{F},P)$ satisfying* H1–H2. *Let V be a convex real valued mapping satisfying* (3.1)–(3.3). *Assume further that $v(C) < \infty$. Then*,

$$v(C) = v(C^{00}) := \inf_{Y\in C^{00},\ E(|V(Y)|)<\infty} E[V(Y)].$$

Proof. Since $C \subset C^{00}$, one has $v(C) \geqslant v(C^{00})$. Let us prove the converse inequality.

Consider some $Y_* \in C^{00}$ such that $E[|V(Y_*)|] < \infty$. By Theorem 2.2 and Remark 2.3, there exists a sequence $(Y_n)_n$ in C and some $Y \in C^{00}$ such that $Y_n \to Y, P$-a.s. and $Y \succcurlyeq_K Y_*$.

Fix some $\bar{Y} \in C$ such that $E[|V(\bar{Y})|] < \infty$. Its existence is guaranteed by the assumption that $v(C) < \infty$.

Choose $\lambda \in (0,1)$. One has $\lambda\bar{Y} + (1-\lambda)Y_n \succcurlyeq_K \lambda\bar{Y}$, and so by (3.2), $V(\lambda\bar{Y} + (1-\lambda)Y_n) \leqslant V(\lambda\bar{Y})$. Now, $V(\lambda\bar{Y}) = V(\lambda\bar{Y})\mathbb{1}_{\ell_K(\bar{Y})\leqslant b} + V(\lambda\bar{Y})\mathbb{1}_{\ell_K(\bar{Y})>b}$. As $\lambda\bar{Y} \succcurlyeq_K \ell_K(\lambda\bar{Y})e = \lambda\ell_K(\bar{Y})e$ by Property 2.4 (iii) and (vi), one has

$$V(\lambda\bar{Y})\mathbb{1}_{\ell_K(\bar{Y})>b} \leqslant V(\lambda be)\mathbb{1}_{\ell_K(\bar{Y})>b}$$

and so, using (3.3),

$$V(\lambda\bar{Y})^+\mathbb{1}_{\ell_K(\bar{Y})>b} \leqslant V(\lambda be)^+\mathbb{1}_{\ell_K(\bar{Y})>b} \leqslant \lambda^{-\beta}V(be)^+\mathbb{1}_{\ell_K(\bar{Y})>b},$$

since, by Property 2.4 (ii) and the normalization $\|e\|_e = 1$, we have $\ell_K(e) \leqslant 1$ which implies that $\ell_K(be) \leqslant b$ by Property 2.4 (iii).

On the other hand, using (3.3) again, one has $V(\lambda\bar{Y})^+\mathbb{1}_{\ell_K(\bar{Y})\leqslant b} \leqslant \lambda^{-\beta}V(\bar{Y})^+\mathbb{1}_{\ell_K(\bar{Y})\leqslant b}$. We conclude that

$$V((1-\lambda)Y_n + \lambda\bar{Y})^+ \leqslant \lambda^{-\beta}[V(be)^+ + V(\bar{Y})^+].$$

It follows that $\{V\left((1-\lambda)Y_n+\lambda\bar{Y}\right)^+,\ n\geqslant 1\}$ is uniformly integrable. Therefore, by (3.2) and the inequalities $(1-\lambda)Y+\lambda\bar{Y}\succcurlyeq_K(1-\lambda)Y_*+\lambda\bar{Y}\succcurlyeq_K(1-\lambda)Y_*$ for all $\lambda\in(0,1)$, using Fatou's lemma and the a.s. continuity of V at $(1-\lambda)Y+\lambda\bar{Y}$,

$$\limsup_{n\to\infty} EV((1-\lambda)Y_n+\lambda\bar{Y})\leqslant EV((1-\lambda)Y+\lambda\bar{Y})\leqslant EV((1-\lambda)Y_*+\lambda\bar{Y})$$

$$\leqslant EV((1-\lambda)Y_*).$$

Since by convexity of $C, (1-\lambda)Y_n+\lambda\bar{Y}\in C$ for each n, it follows that, for all $\lambda\in(0,1)$,

$$\inf_{Z\in C} EV(Z)\leqslant EV((1-\lambda)Y_*).$$

Using (3.3) again, we see that $\{V((1-\lambda)Y_*)^+,\ \lambda\in(0,1/2)\}$ is uniformly integrable. Letting $\lambda\to 0$, we therefore obtain that

$$\inf_{Z\in C} EV(Z)\leqslant EV(Y_*).$$

This shows that $v(C)\leqslant v(C^{00})$. □

Remark 3.2. This result extends Proposition 3.2 of Kramkov and Schachermayer (1999), in the sense that we consider a multivariate framework and do not assume that C is closed under countable convex combinations. However, the above arguments heavily rely on the assumption that $v(C)<\infty$. It cannot be used to retrieve the implication $v(C^{00})<\infty\Rightarrow v(C)<\infty$. In some specific cases, for instance if $V(y)$ is of the form $\phi(\ell_K(y))$ or $\phi(x\cdot y)$ for some convex decreasing function ϕ and some $x\in K^0$, it is possible to follow the approximation argument of Kramkov and Schachermayer (1999) but this appears to be complicated in the general case.

In order to obtain an existence result in C^{00}, we shall appeal to one of the following conditions

V1: $V\geqslant M$ for some $M\in\mathbb{R}$.
V2: V is not bounded from below, $r\in\mathbb{R}_+\mapsto V(re)$ is strictly convex for some $e\in\mathrm{ri}(K)$ and $\lim_{r\to+\infty}V'(re)=0$ (where V' denotes the right-hand derivative of V).

Observe that, up to a normalization, we can assume that $\|e\|=1$.

Proposition 3.3. *Let C be a convex subset of $L^0(K;\Omega,\mathscr{F},P)$ satisfying* H1–H2 *such that $C^0\cap[0,\infty)^d\setminus\{0\}\neq\emptyset$. Let V be a convex real valued mapping satisfying* (3.1)–(3.2) *such that either* V1 *or* V2 *hold. Assume that $v(C^{00})<\infty$. Then, there exists some $Y_*\in C^{00}$ such that*

$$v(C^{00})=E[V(Y_*)].$$

Proof. (1) Let $(Y_n)_n$ be a minimizing sequence for $v(C^{00})$. C^{00} is bounded in $L^1(\mathbb{R}^d;\Omega,\mathscr{F},P)$ since there is some non zero constant α with non-negative components in C^0. Suppose indeed that $\alpha^j>0$. As $\{y\in K, y^j=1\}$ is compact, there is a constant $\lambda>0$ such that for $y\in K,\ y\leqslant\lambda|y^j|$. Therefore, if $Y\in C^{00}$,

$$E(\|Y\|) \leqslant \lambda E(Y^j) = \frac{\lambda}{\alpha^j}\, E(Y^j \alpha^j) \leqslant \frac{\lambda}{\alpha^j}\, E(Y \cdot \alpha) \leqslant \frac{\lambda}{\alpha^j}.$$

Using the convexity of K, we deduce from Komlòs's Lemma (Komlòs, 1967) that there is a subsequence $(\tilde{Y}_n)_n$ with $\tilde{Y}_n \in \operatorname{conv}\{Y_k,\, k \geqslant n\}$ such that $\tilde{Y}_n \to Y_*$ P-a.s. for some $Y_* \in L^0(K; \Omega, \mathscr{F}, P)$. By the convexity of C^{00}, $\tilde{Y}_n \in C^{00}$ so that by closedness of C^{00} (Theorem 2.2), $Y_* \in C^{00}$. We shall prove in step 2 that

$$\text{the sequence } (V(\tilde{Y}_n)^-)_n \text{ is uniformly integrable.} \tag{3.6}$$

V being convex, $(\tilde{Y}_n)_n$ is also a minimizing sequence, so that by (3.6) and Fatou's Lemma

$$v(C^{00}) = \liminf_{n\to\infty} E[V(\tilde{Y}_n)] \geqslant E[V(Y_*)].$$

Since $Y_* \in C^{00}$, this proves the desired result.

(2) We now prove (3.6). If V1 holds, V is bounded from below, so that (3.6) holds. We now assume that V2 holds. As $e \in \operatorname{ri}(K)$, for all $y \in K$, we can find some $r(y) > 0$ such that $r(y)e \succcurlyeq_K y$.

It follows from the fact that V is K-decreasing and V^- is unbounded that

$$\liminf_{r\to\infty} V(re) = -\infty. \tag{3.7}$$

Let ϕ be the inverse of $r \in (0,\infty) \mapsto -V(re)$ so that ϕ is well defined and is convex by V2. Observe that ϕ is non decreasing by (3.2). Then, by Property 2.4 (vi) and (3.2)

$$\begin{aligned} E[\phi(V(\tilde{Y}_n)^-)] &\leqslant E\left[\phi\left(V\left(-\frac{\ell_K(-\tilde{Y}_n)}{\ell_K(e)}e\right)^-\right)\right] \\ &\leqslant \phi(0) + E\left[-\frac{\ell_K(-\tilde{Y}_n)}{\ell_K(e)}\right], \end{aligned}$$

where the last inequality comes from separating the cases where $V\left(-\ell_K(-\tilde{Y}_n)e/\ell_K(e)\right)$ is positive or negative. The last term is uniformly bounded since (Y_n) is bounded in $L^1(\mathbb{R}^d; \Omega, \mathscr{F}, P)$ and

$$-\ell_K(-\tilde{Y}_n) = \max_{x\in\tilde{K}_1^0} x \cdot \tilde{Y}_n \leqslant \sum_{i=1}^d \frac{1}{e^i}\, \tilde{Y}_n^i.$$

Finally, $E[\phi(V(\tilde{Y}_n)^-)]$ is uniformly bounded. To prove (3.6), it remains to prove that

$$\lim_{r\to\infty} \frac{\phi(r)}{r} = \infty. \tag{3.8}$$

Observe that

$$\liminf_{r\to\infty} \frac{\phi(r)}{r} = \liminf_{r\to\infty} \frac{r}{-V(re)}.$$

By L'Hôpital rule, as $\lim_{r\to+\infty} V'(re)=0$,

$$\liminf_{r\to\infty} \frac{r}{-V(re)} = \infty$$

which proves (3.8). □

3.2. Application to the dual formulation of the utility maximization problem under proportional transaction costs

We now turn to the application of Proposition 3.1 to the problem of utility maximization in financial markets with proportional transaction costs. When the underlying financial assets are modelled by diffusion processes, this problem can be studied by analytical methods (see e.g. the pioneering work of Davis and Norman (1990)). Here, we shall consider the duality approach which was first used, in this context, by Cvitanić and Karatzas (1996). Our aim is not to obtain an existence or duality result but only to show how Theorem 2.2 and Proposition 3.1 may be used in order to extend the result of Theorem 2.2 (iv) of Kramkov and Schachermayer (1999), to our context. For the convenience of the reader, we first describe the financial market and the convex subset (of $L^0(\mathbb{R}^d;\Omega,\mathscr{F},P)$) of natural dual variables. We then consider its polar and bipolar sets and explain how they are related to financial strategies. Finally, we introduce the associate dual problems and provide conditions under which they lead to the same value function, in the spirit of Theorem 2.2 (iv) of Kramkov and Schachermayer (1999). This is the object of Corollary 3.8 and Lemma 3.9.

3.2.1. The financial market

Given a finite time horizon T, we consider a complete probability space, with trivial initial σ-algebra, $(\Omega,\mathscr{F},\mathbb{F}=(\mathscr{F}_t)_{t\leqslant qT},P)$, supporting a semimartingale $S:=(S^1,\dots,S^d)$ with positive components: $S^i>0\,P$-a.s. for each $i=1,\dots,d$. Here, S^i will be interpreted as the price process corresponding to the ith asset.

We next define the closed convex cone of $[0,\infty)^d$

$$K=\{y\in[0,\infty)^d : y^j-(1+\lambda^{ij})y^i\leqslant 0,\quad 1\leqslant i,j\leqslant d\},$$

for some $\lambda\in\mathbb{M}^d_+$, the set of square matrices with d columns and non-negative entries. We shall later interpret λ^{ij} as the proportional transaction cost which is paid when one transfers money from the account invested in the ith asset to the account invested in the jth asset.

3.2.2. Hedgeable contingent claims and duality

For each $z\in K$, we define the convex subset of $L^0(K;\Omega,\mathscr{F}_T,P)$

$$C(z):=\{Z_T,\ Z\in\mathscr{D} \text{ with } Z_0=z\},$$

where $\mathscr{D}$ is the set of $\mathbb{F}$-adapted K-valued processes Z such that Z^iS^i is a martingale for all $i\leqslant d$. Observe that $0\in\mathscr{D}\neq\emptyset$.

Let us interpret this set in financial terms. First, we can easily check that the positive polar of K is given by

$$K^0 = \left\{ x \in \mathbb{R}^d : \exists a \in \mathbb{M}^d_+,\ x^i + \sum_{j \neq i} a^{ji} - (1+\lambda^{ij}) a^{ij} \geqslant 0, \quad 1 \leqslant i \leqslant d \right\}$$

(see Kabanov, 1999). Here K^0 has the following interpretation. Consider a financial market where a transfer a^{ij} of money from the account invested in asset i to the one invested in asset j induces a transaction cost $\lambda^{ij} a^{ij}$: to obtain the amount a^{ij} of asset j we have to pay $(1+\lambda^{ij})a^{ij}$ of asset i. Then, the element of K^0 can be interpreted as the vectors of portfolio holdings such that a *no-bankruptcy condition* holds which means that the liquidation value of the portfolio is non-negative: after some suitable (instantaneous) transfers of funds (represented by the matrix a) between the different parts of the portfolio, we can obtain a portfolio $\tilde{x}$, defined by

$$\tilde{x}^i = x^i + \sum_{j \neq i} a^{ji} - (1+\lambda^{ij}) a^{ij} \quad \text{for } 1 \leqslant i \leqslant d,$$

with non-negative components. K^0 is the so-called *solvency region*. Observe that for $x,\ x' \in \mathbb{R}^d$, $x - x' \in K^0$ means that, if we start with the portfolio x, we can find some suitable transfers such that the induced portfolio $\tilde{x}$ dominates x' component by component.

We now fix an initial endowment $x \in \mathrm{Int}(K^0)$ (observe that $(0,\infty)^d \subset \mathrm{Int}(K^0) \neq \emptyset$). A financial strategy is modelled by an adapted process L with values in $\mathbb{M}^d_+$ such that each component L^{ij} is non-decreasing, right-continuous, has bounded variations and satisfies $L^{ij}_{0-} = 0$. For each $t \in [0,T], L^{ij}_t$ represents the cumulative amount of money transferred from the account invested in asset i to the account invested in asset j in the time interval $[0,t]$. We denote by $\mathscr{A}$ the set of such processes. Given $L \in \mathscr{A}$, the induced portfolio process $X^{x,L}$ is defined as the solution of

$$X^i_t = x^i + \int_0^t \left(X^i_{t-} \frac{\mathrm{d}S^i_t}{S^i_{t-}} + \sum_{j \neq i, j=1}^{d} \mathrm{d}L^{ji}_t - (1+\lambda^{ij})\, \mathrm{d}L^{ij}_t \right) \quad i \leqslant d.$$

Here, X^i corresponds to the amount of money invested in the ith asset.

We shall denote by $\mathscr{A}_b$ the set of $L \in \mathscr{A}$ such that, for each $t \in [0,T]$, $X^{x,L}_t - c_L S_t \in K^0$ P-a.s. for some real valued constant c_L.

We define $\mathscr{X}_b(x)$ (resp. $\mathscr{X}^s_b(x)$) as the set of random variables $X \in L^0(K^0; \Omega, \mathscr{F}, P)$ such that, for some $L \in \mathscr{A}_b$, $X^{x,L}_T = X$ P-a.s. (resp. $X^{x,L}_T - X \in K^0$ P-a.s.). This is the set of *attainable* (resp. *super-hedgeable*) contingent claims starting with the initial endowment x.

Lemma 3.4. *For all $Z \in \mathscr{D}$ and $L \in \mathscr{A}_b$, the process $(X^{x,L}_t \cdot Z_t)_{t \leqslant T}$ is a super-martingale. In particular, for all $X \in \mathscr{X}^s_b(x)$ and $(z,Z) \in K \times C(z)$*

$$E[X \cdot Z] \leqslant x \cdot z. \tag{3.9}$$

The proof of the super-martingale property can be found in Kabanov and Striker (2002). Inequality (3.9) is then a direct consequence of the definition of $C(z)$ and $\mathscr{X}_b^s(x)$.

Remark 3.5. Under some mild assumptions on the model, $\mathscr{X}_b^s(x)$ can be completely characterized by $\mathscr{D}$ (equivalently $\{(z,Z)\in K\times C(z)\}$) through the *hedging theorem*

$$\mathscr{X}_b^s(x)=\{X\in L^0(K^0;\Omega,\mathscr{F},P):E[X\cdot Z_T-x\cdot Z_0]\leqslant 0 \text{ for all } Z\in\mathscr{D}\},$$

$$=\{X\in L^0(K^0;\Omega,\mathscr{F},P):E[X\cdot Z]\leqslant x\cdot z \text{ for all } (z,Z)\in K\times C(z)\}.$$

This provides a dual characterization of $\mathscr{X}_b^s(x)$ in terms of $\{(z,Z)\in K\times C(z)\}$. We shall not enter into details here and we refer to Kabanov and Striker (2002), for conditions under which this equality holds.

The above lemma together with Remark 3.5 show that the elements of $\mathscr{D}$ play a similar role as the density of martingale measures for S in frictionless markets : the product of a portfolio with an element of $\mathscr{D}$ is a super-martingale, and, under some mild assumptions, the set of *super-hedgeable* claims is completely characterized by $\mathscr{D}$ through the *hedging theorem*.

It follows that there is a natural duality between $\mathscr{X}_b^s(x)$ and $\{(z,Z)\in K\times C(z)\}$, which can be exploited to prove that existence holds in the utility maximization problem, as described in the next subsection.

For each $z\in K$, we now define the "positive" polar and bipolar of $C(z)$

$$C^0(z):=\left\{X\in L^0(K^0;\Omega,\mathscr{F},P):E\left[X\cdot Z\right]\leqslant 1 \text{ for all } Z\in C(z)\right\},$$

$$C^{00}(z):=\left\{Z\in L^0(K;\Omega,\mathscr{F},P):E\left[X\cdot Z\right]\leqslant 1 \text{ for all } X\in C^0(z)\right\}.$$

The next result shows that (3.9) still holds if we replace $C(z)$ by $C^{00}(z)$.

Property 3.6. *For all* $X\in\mathscr{X}_b^s(x)$ *and* $(z,Z)\in K\times C^{00}(z)$,

$$E[X\cdot Z]\leqslant x\cdot z. \tag{3.10}$$

Proof. Fix $z\in K$ and $X\in\mathscr{X}_b^s(x)$. If $z\neq 0$, then $x\cdot z>0$ since we have assumed that $x\in\mathrm{Int}(K^0)$. It then follows from (3.9) that $1/x\cdot z X\in C^0(z)$ which proves the result. If $z=0$, then, by definition of $\mathscr{D}, E[S_T^iZ^i]=0\,P$-a.s. for each $Z\in C(z)$ and $i\in\{1,\ldots,d\}$. Since $S_T^i>0$ for each $i\in\{1,\ldots,d\}$ by assumption and $K\subset[0,\infty)^d$, this shows that $Z=0\,P$-a.s. so that (3.10) still holds. □

3.2.3. Natural duality in the utility maximization problem

Observe that, if the characterization in Remark 3.5 holds, using Property 3.6 one has

$$\mathscr{X}_b^s(x)=\bigcap_{z\in K}\{X\in L^0(K^0;\Omega,\mathscr{F},P):E[X\cdot Z]\leqslant x\cdot z \text{ for all } Z\in C^{00}(z)\}. \tag{3.11}$$

We are interested by the following control problem:

$$u(\mathscr{X}_b(x)):=\sup_{X\in\mathscr{X}_b(x)}E[U(X)],$$

where U is a K^0-increasing, concave, real-valued mapping with $\mathrm{Int}(K^0) \subset \mathrm{dom}(U) \subset K^0$. Here, U is interpreted as a *utility function* and $u(\mathscr{X}(x))$ is the maximal *expected utility* that the financial agent can reach with the initial holding x. In order to exclude trivial cases, we shall assume that

$$u(\mathscr{X}_b(x)) > -\infty. \tag{3.12}$$

Letting V denote the Fenchel transform

$$V(y) := \sup_{x \in K^0} [U(x) - x \cdot y],$$

we deduce from (3.9) that

$$u(\mathscr{X}_b(x)) \leqslant \inf_{z \in K} v(C(z)) + z \cdot x, \quad \text{where, for } z \in K,$$

$$v(C(z)) := \inf_{Z \in C(z)} E[V(Z)]. \tag{3.13}$$

Hence, because of Remark 3.5, the dual problem is naturally defined as

$$\inf_{z \in K} \inf_{Z \in C(z)} E[V(Z)] + z \cdot x.$$

However, as already mentioned in the above subsection, the optimum may not be attained in $\{(z,Z) \in K \times C(z)\}$. It is therefore natural to extend the set of dual variables and to consider

$$v(C^{00}(z)) := \inf_{Z \in C^{00}(z)} E[V(Z)]. \tag{3.14}$$

Observe that, by (3.12), (3.10) and the inclusion $C(z) \subset C^{00}(z)$, we have

$$-\infty < u(\mathscr{X}_b(x)) \leqslant \inf_{z \in K} v(C^{00}(z)) + z \cdot x \leqslant \inf_{z \in K} v(C(z)) + z \cdot x. \tag{3.15}$$

It is generally easier to prove existence of a solution to the dual problem than to the primal one. This also usually allows a characterization the optimal solution of the primal problem. One first proves the existence of some $(z^*, Z^*) \in K \times C^{00}(z^*)$ such that

$$E[V(Z^*)] + z^* \cdot x = \inf_{z \in K} \inf_{Z \in C^{00}(z)} E[V(Z)] + z \cdot x.$$

Remark 3.7. This existence result can be obtained by using similar arguments as in the proof of Proposition 3.3 if $C^0(z) \cap [0,\infty)^d \setminus \{0\} \neq \emptyset$ for all $z \in K$. If we assume that S^1 represents the bond process in the market and if we use the usual normalization $S^1 \equiv 1$, i.e. if we think in terms of discounted prices, then for all $x \in \mathrm{Int}(K^0)$, we can find some $\tilde{x} = (\tilde{x}^1, 0, \ldots, 0) \in \mathscr{X}_b(x)$ with $\tilde{x}^1 > 0$. Here, $\tilde{x}^1$ is defined as the argmax of $\{w > 0 : \ x - (w, 0, \ldots, 0) \in K^0\}$ which is positive since $x \in \mathrm{Int}(K^0)$ (see Bouchard (2002), for a detailed presentation of the related notion of *liquidating function*). Since, by Lemma 3.4, $\mathscr{X}_b(x) \subset C^0(z)$ for all $z \in K$, it follows that $\tilde{x} \in C^0(z)$ for all $z \in K$ therefore $C^0(z) \cap [0,\infty)^d \setminus \{0\} \neq \emptyset$ for all $z \in K$.

Assuming that there exists some $X^* \in -\partial V(Z^*)(\subset K^0)$ for which

$$\sup_{(z,Z)\in K\times C^{00}(z)} E[Z\cdot X^* - z\cdot x] = E[Z^*\cdot X^* - z^*\cdot x] = 0,$$

we obtain that

$$E[U(X^*)] = E[V(Z^*) + Z^*\cdot X^*] = E[V(Z^*)] + z^*\cdot x = \inf_{z\in K} v(C^{00}(z)) + z\cdot x,$$

where, under (3.11), $X^* \in \mathscr{X}_b^s(x)$. By (3.15), this shows that

$$u(\mathscr{X}_b(x)) = E[U(X^*)] = \inf_{z\in K} v(C^{00}(z)) + z\cdot x,$$

and that $X^* \in \mathscr{X}_b(x)$ since U is K^0-increasing. The existence of some X^* satisfying the above conditions is obtained by means of calculus of variation technics on the dual problem (we refer to e.g. Davis and Norman (1990), Cvitanić and Karatzas (1996), Kabanov (1999), Cvitanić, and Wang (2001), Bouchard (2002) or Deelstra et al. (2002), for detailed proofs of existence and duality results which are based on this approach).

This leads to the usual duality result:

$$u(\mathscr{X}_b(x)) = \inf_{z\in K} [v(C^{00}(z)) + z\cdot x]. \tag{3.16}$$

However, this duality is only obtained in terms of $C^{00}(z)$, where $C^{00}(z)$ is constructed in an artificial way. It is natural to ask whether it still holds if we consider the natural set of dual variables $C(z)$. Proposition 3.1 allows one to obtain a partial answer.

Clearly, K satisfies H1. Using Property 2.4, it is also easily checked that V satisfies (3.2). Moreover, by Lemma 4.2 in Deelstra et al. (2002), (3.1) holds as soon as (3.4) does. Hence, we may apply Proposition 3.1, and obtain the extension of Theorem 2.2 (iv) of Kramkov and Schachermayer (1999), stated in the context of financial markets without transaction costs:

Corollary 3.8. *Assume that V satisfies (3.4). Let $z\in K$ be such that* H2 *holds for $C(z)$ and $v(C(z)) < \infty$. Then,*

$$v(C^{00}(z)) = v(C(z)).$$

It is possible that $C(z)$ does not satisfy H2 for all $z\in K$. We therefore need an extra assumption to prove that equality holds between the two right hand-side terms of (3.15).

Proposition 3.9. *Assume that V satisfies (3.4) and that $\inf_{z\in K} v(C(z)) < \infty$.*

(i) *If*

$$\inf_{z\in K} [v(C^{00}(z)) + z\cdot x] = \inf_{z\in K_{\mathrm{H2}}} [v(C^{00}(z)) + z\cdot x] \tag{3.17}$$

where $K_{\mathrm{H2}} := \{z\in K : C(z)\cap L^0(\mathrm{ri}K;\Omega,\mathscr{F},P) \neq \emptyset\}$, then

$$\inf_{z\in K} [v(C^{00}(z)) + z\cdot x] = \inf_{z\in K} [v(C(z)) + z\cdot x].$$

(ii) *If $\mathrm{ri}(K) \subset K_{\mathrm{H2}}$, then (3.17) holds.*

Remark 3.10. Assume that there exists some $Q \sim P$ such that S is a Q-martingale. Set $H_T = E[dQ/dP|\mathscr{F}_T]$. Then, for all $\bar{z} \in \mathrm{ri}(K)$, $(\bar{Z}_t)_{t \leqslant T} = (\bar{z}E[H_T|\mathscr{F}_t])_{t \leqslant T}$ lies in $\mathscr{D}$ and satisfies $\bar{Z}_T \in C(\bar{z}) \cap L^0(\mathrm{ri}(K); \Omega, \mathscr{F}, P)$. It follows that H2 holds for all $\bar{z} \in \mathrm{ri}(K)$, i.e. $\mathrm{ri}(K) \subset K_{\mathbf{H2}}$.

Proof. We shall use the same arguments as in the proof of Proposition 3.1.

(i) We fix $(\bar{z}, \bar{Z}) \in K \times C(\bar{z})$ such that $E[V(\bar{Z})] < \infty$ and some $(z_*, Z_*) \in K_{\mathbf{H2}} \times C^{00}(z_*)$ such that $E[V(Z_*)] < \infty$. Since H2 holds for $C(z_*)$, we can find a sequence $(Z^n)_n \in C(z_*)$ and some $Z \in C^{00}(z_*)$ such that, up to a subsequence, $Z^n \to Z$ P-a.s. and $Z \succcurlyeq_K Z_*$ P-a.s. For $\lambda \in (0,1)$, we deduce from the same arguments as in the proof of Proposition 3.1 that

$$\limsup_{n\to\infty} E[V((1-\lambda)Z^n + \lambda\bar{Z})] \leqslant E[V((1-\lambda)Z_*)].$$

Since $(1-\lambda)Z^n + \lambda\bar{Z} \in C((1-\lambda)z_* + \lambda\bar{z})$, we deduce that

$$\inf_{z\in K} [v(C(z)) + z \cdot x] \leqslant E[V((1-\lambda)Z_*)] + (1-\lambda)z_* \cdot x + \lambda\bar{z} \cdot x.$$

Finally, letting $\lambda \to 0$ and arguing as in the proof of Proposition 3.1, we obtain

$$\inf_{z\in K} [v(C(z)) + z \cdot x] \leqslant \inf_{z\in K_{\mathrm{H2}}} [v(C^{00}(z)) + z \cdot x] := \inf_{z\in K} v(C^{00}(z)) + z \cdot x$$

(the last equality holds by assumption).

(ii) Fix $(z_*, Z_*) \in K \times C^{00}(z_*)$ such that $E[V(Z_*)] < \infty$, $\bar{z} \in \mathrm{ri}(K)$ and $\bar{Z} \in C^{00}(\bar{z})$. Observe that for $\lambda \in (0,1)$, $(1-\lambda)Z_* + \lambda\bar{Z} \in C^{00}((1-\lambda)z_* + \lambda\bar{z})$ and $(1-\lambda)z_* + \lambda\bar{z} \in \mathrm{ri}(K) \subset K_{\mathrm{H2}}$ by assumption. Using the fact that V is K-non decreasing, this shows that

$$\begin{aligned}\inf_{z\in K_{\mathbf{H2}}} [v(C^{00}(z)) + z \cdot x] &\leqslant E[V((1-\lambda)Z_* + \lambda\bar{Z})] + ((1-\lambda)z_* + \lambda\bar{z}) \cdot x \\ &\leqslant E[V((1-\lambda)Z_*)] + ((1-\lambda)z_* + \lambda\bar{z}) \cdot x,\end{aligned}$$

which proves that

$$\inf_{z\in K_{\mathbf{H2}}} [v(C^{00}(z)) + z \cdot x] \leqslant \inf_{z\in K}[v(C^{00}(z)) + z \cdot x],$$

as in (i). $\square$

Acknowledgements

The authors wish to thank the three anonymous referees for their careful reading of the first version of the paper and numerous useful comments.

Appendix A

To be complete, we give below the proof of the elementary Property 2.4.

(i) is a direct consequence of the fact that for $y \in K$, $y^i \geqslant 0$ for each $1 \leqslant i \leqslant d$.

(ii) By (i), e belongs to $\tilde{K}_1^0$, so that $\ell_K(y) \leqslant \|y\|_e$.

(iii) The homogeneity is obvious. The continuity is easily deduced from the fact that $\tilde{K}_1^0$ is compact.

(iv) Obviously, if $y \in K$ then $\ell_K(y) \geqslant 0$. Suppose now that the previous inequality holds and that $y \notin K$. By Hahn–Banach theorem, one may find an $x \in \mathbb{R}^d$ such that for any $z \in K$, $x \cdot z \geqslant 0$ and $x \cdot y < 0$. Let $\pi(x)$ denote the orthogonal projection of x on $\mathscr{K}$. Then, $(x - \pi(x)) \cdot z = 0$ for all $z \in \mathscr{K}$ so that, for any $z \in K, \pi(x) \cdot z \geqslant 0$ (and so $\pi(x) \in \tilde{K}^0$) and as $y \in \mathscr{K}$, $\pi(x) \cdot y = x \cdot y < 0$. If $\|\pi(x)\|_e = 0$, we get a contradiction. On the other hand, if $\|\pi(x)\|_e > 0$, then $\pi(x)/\|\pi(x)\|_e \in \tilde{K}_1^0$, and so $\pi(x)/\|\pi(x)\|_e \cdot y \geqslant \ell_K(y) \geqslant 0$, a contradiction too.

(v) Suppose that $y \in \mathrm{ri}(K)$. There exists $\varepsilon > 0$ such that if $\|x - y\| < \varepsilon$ and $x \in \mathscr{K}$, then $x \in K$. Observe that there is a $\gamma > 0$ such that for any x, $\gamma \|x\| \leqslant \|x_e\|$. Suppose $\ell_K(y) = 0$. One may find $x^* \in \tilde{K}_1^0 \subset \mathscr{K}$ such that $x^* \cdot y \leqslant \varepsilon/2\gamma^2$. For any $-\varepsilon < \lambda < 0$, $y + \lambda x^* \in K$. Therefore, $0 \leqslant x^* \cdot (y + \lambda x^*) = x^* \cdot y + \lambda \|x^*\|^2 \leqslant \varepsilon/2\gamma^2 + \lambda/\gamma^2$, using in the last inequality the fact that $\|x^*\|_e = 1$. Hence a contradiction, if $\lambda < -\varepsilon/2$.

The converse implication results from the continuity of ℓ_K and the characterization (iv) of K.

(vi) For $x \in \tilde{K}_1^0$, one has $x \cdot (y - \ell_K(y)e) \geqslant x \cdot y - \ell_K(y)\|x\|_e = x \cdot y - \ell_K(y) \geqslant 0$ so that $y \succcurlyeq_K \ell_K(y)e$.

Moreover, $-\ell_K(-y)e - y\ell_K(e) = \max_{z \in \tilde{K}_1^0} z \cdot ye - y\ell_K(e)$. Therefore, $x \cdot (\max_{z \in \tilde{K}_1^0} z \cdot ye - y\ell_K(e)) = [\max_{z \in \tilde{K}_1^0} z \cdot y] x \cdot e - x \cdot y \ell_K(e) \geqslant \ell_K(e)[\max_{z \in \tilde{K}_1^0} z \cdot y - x \cdot y] \geqslant 0$ for all $x \in \tilde{K}_1^0$.

(vii) For $x \in \tilde{K}_1^0$, using (v), one has $0 \leqslant x \cdot (y - fe) \leqslant x \cdot y - fc_e$ for some $c_e > 0$ since $e \in \mathrm{ri}(K)$. $\square$

References

Bouchard, B., 2002. Utility maximization on the real line under proportional transaction costs. Finance Stochast. 6, 4.

Brannath, W., Schachermayer, W., 1999. A Bipolar Theorem for $L_+^0(\Omega, \mathscr{F}, \mathbb{P})$. Séminaire de Probabilités, Vol. XXX, pp. 349–354.

Cvitanić, J., Karatzas, I., 1996. Hedging and portfolio optimization under transaction costs: a martingale approach. Math. Finance 6, 133–165.

Cvitanić, J., Wang, H., 2001. On optimal terminal wealth under transaction costs. J. Math. Econom. 35 (2), 223–231.

Davis, M., Norman, A., 1990. Portfolio selection with transaction costs. Math. Oper. Res. 15, 676–713.

Deelstra, G., Pham, H., Touzi, N., 2002. Dual formulation of the utility maximization problem under transaction costs. Ann. Appl. Probab. 11 (4), 1353–1383.

Delbaen, F., Schachermayer, W., 1999. A general version of the fundamental theorem of asset pricing. Math. Ann. 300, 463–520.

Kabanov, Yu., 1999. Hedging and liquidation under transaction costs in currency markets. Finance Stochast. 3, 237–248.

Kabanov, Yu., Striker, Ch., 2002. Hedging of contingent claims under transaction costs. Preprint.

Komlòs, J., 1967. A generalisation of a theorem of Steinhaus. Acta Math. Acad. Sci. Hungar. 18, 217–229.

Kramkov, D., Schachermayer, W., 1999. The asymptotic elasticity of utility functions and optimal investment in incomplete markets. Ann. Appl. Probab. 9 (3), 904–950.

Schachermayer, W., 2000. Optimal investment in incomplete financial markets. In: Geman, H., Madan, D., Pliska, St.R., Vorst, T. (Eds.), Bachelier Congress, pp. 427–462.

Schaefer, H.H., 1996. Topological vector spaces. Graduate Texts in Mathematics. Springer, Berlin.